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The Fractal Interpretation of the Weak Scattering of Elastic Waves

Bruce J. West[1,2,3] and Michael F. Shlesinger[4]

Nondestructive methods, in particular the measurement of elastic waves, have become increasingly important in determining the microstructure of many materials in recent years. One of these methods is observing the attenuation of ultrasonic waves of known amplitude and direction, e.g., in granular metals. The waves are exponentially attenuated with distance with a frequency-dependent attenuation fractor. The attenuation factor can be decomposed into two parts: absorption and scattering. Experimentally, the absorption part varies linearly with frequency, while the scattering part has a noninteger power law behavior, the exponent of which is related to the strength of the material. Theoretically, at long wavelengths the exponent is 4 (Rayleigh scattering) while for grain-sized wavelengths it is 2 (diffusive scattering). We relate the attenuation factor to the forward scattering amplitude which is related to the frequency dependence of the scatterers and their cross sections. We attribute the noninteger attenuation exponent to a fractal distribution of grain shapes and sizes.

KEY WORDS: Random media; fractals; elastic waves; nondestructive evaluation; ultrasonics.

1. INTRODUCTION

The study of the propagation of waves in random media, and in particular the effects of multiple scattering on the wave traversing the media, had its first success with the investigation of Foldy.[1] He was interested in such processes as the multiple scattering of a sound wave by the water droplets of a fog where interference effects may not be neglected. He considered the first and second moment properties of scalar waves traveling in a medium of

[1] Center for Studies of Nonlinear Dynamics, La Jolla Institute, 8950 Villa La Jolla Drive, Suite 2150, La Jolla, California 92037.
[2] Supported by the Defense Advanced Research Projects Agency.
[3] Affiliated with the University of California, San Diego.
[4] Office of Naval Research, Physics Division, 800 N. Quincy St., Arlington, Virginia 22217.

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randomly distributed, isotropic scatterers, and calculated the index of refraction. Lax[2] later generalized this self-consistent treatment to include such effects as inelastic scattering, partial or complete ordering of the scatterers in addition to random scatterers and the effects of anisotropy. He also considered the quantum mechanical determination of the exact index of refraction in crystals containing defects, i.e., scattering centers. The index of refraction is the important physical quantity because it determines both the wavelength and the rate of attenuation of the propagating wave in the scattering region.

More recently these considerations have been extended to the study of multiple scattering of ultrasonic waves in elastic media with randomly positioned scatterers.[3,4] The motivation for these latter studies is the use of ultrasonic waves as probes of the material. Material properties such as the distribution of grain size in polycrystalline materials, the degree of homogeneity, the existence of macroscopic cracks, inclusions, twin boundaries, dislocations, etc. all affect fracture micromechanisms and fracture control technology. The basis of the ultrasonic approach is the observation that the amplitude of low-frequency (long-wavelength) ultrasonic waves (of known amplitude and direction) are exponentially attenuated with distance, i.e., the wave intensity decays as $\exp[-\alpha(\omega)z]$, where z is the line-of-sight distance between the source and receiver and $\alpha(\omega)$ is a frequency-dependent attenuation factor.

The standard theories of wave attenuation partition $\alpha(\omega)$ into two parts; the absorption attenuation coefficient $\alpha_a(\omega)$ and the scattering attenuation coefficient $\alpha_s(\omega)$.[5] The absorption of energy from the ultrasonic wave arises from dislocation damping as well as magneto-elastic and thermo-elastic hysteresis and leads to the coefficient

$$\alpha_a(\omega) = A\omega \tag{1.1}$$

The discontinuity in acoustic impedance occurring at grain boundaries and defect sites, on the other hand, leads to elastic scattering. For a polycrystalline material of average grain size D an ultrasonic wave of wavelength λ and frequency ω will have the scattering attenuation coefficient: for $\lambda \gg D$ (Rayleigh scattering domain)

$$\alpha_s(f) = S_1 D^3 \omega^4 \tag{1.2}$$

for $\lambda \leqslant D$ (stochastic scattering domain)

$$\alpha_s(f) = S_2 D \omega^2 \tag{1.3}$$

and for $\lambda \ll D$ (diffusive domain)

$$\alpha_s = S_3/D \tag{1.4}$$

where S_1, S_2, and S_3 are related to the elastic moduli and the longitudinal and transverse sound speeds in a single crystal. Experimental data are fitted by (1.2)–(1.4) and the characteristic value of D is inferred.

In typical metals one encounters a range of grain sizes so that neither the Rayleigh nor stochastic scattering limits are appropriate. Instead both types of scattering can be simultaneously present. In addition the absorption parameter A may be weakly dependent on frequency. Thus none of the attenuation coefficients (1.1)–(1.4) are used in practice; instead the phenomenological expression

$$\alpha(\omega) = B\omega^{\mu}, \qquad 1 \leqslant \mu \leqslant 4 \tag{1.5}$$

has been adopted, where B and μ are frequency-independent constants. It is expected, however, that variations in the microstructure of the materials will lead to variations in B and μ, so that these parameters can be expressed in terms of certain structural properties of the material. We show below that the parameter μ is a measure of the density of scatterers in the material.[6]

The nonintegral value of μ in (1.5) suggests that the volume of space occupied by the scatterers is also nonintegral. We argue in Section 3 that the volume occupied by the scatterers is fractal and that μ is a direct measure of the dimensionality of this "volume." To construct this argument we review in Section 2 the theory of weak multiple scattering of an elastic wave in a random media and determine an analytic expression for the complex index of refraction.

2. ELASTIC WAVES

Consider a small amplitude disturbance in an elastic solid $\xi(x, t)$. The vector $\xi(x, t)$ is the displacement of a physical point at the space time location (x, t) in the solid away from its equilibrium position, i.e., in equilibrium the homogeneous solid is free of stress and strain. From the conservation of momentum equation and in the absence of any body force the displacement field in a harmonic solid satisfies the equation of motion

$$\ddot{\xi}(x, t) = \frac{\mu^0}{\rho^0} \nabla^2 \xi(x, t) + \frac{\lambda^0 + \mu^0}{\rho^0} \nabla(\nabla \cdot \xi(x, t)) \tag{2.1}$$

where λ^0 and μ^0 are the Lamé parameters of the homogeneous medium and ρ^0 is the uniform density. The homogeneous equation (2.1) describes the

propagation of a vector displacement in an elastic solid free of defects. In a real material the density and Lamé parameters may vary owing to the presence of grain boundaries, cracks, flaws, inclusions, voids, etc. We indicate the inhomogeneous properties by

$$\rho = \rho^0 + \delta\rho, \qquad \lambda = \lambda^0 + \delta\lambda, \qquad \mu = \mu^0 + \delta\mu \tag{2.2}$$

where $\delta\rho$, $\delta\lambda$, and $\delta\mu$ indicate the allowed inhomogeneities.

For a harmonic disturbance of frequency ω, i.e., $\xi(x, t) = \eta(x)\, e^{-i\omega t}$, the inhomogeneous equation of motion replacing (2.1) is

$$\omega^2\eta(x) + \frac{\mu^0}{\rho^0}\nabla^2\eta(x) + \frac{\lambda^0 + \mu^0}{\rho^0}\nabla(\nabla \cdot \eta(x)) = -\omega^2\frac{\delta\rho}{\rho^0}\eta(x) + \nabla \cdot \delta P \tag{2.3}$$

where the variation in the stress tensor P is

$$\delta P_{ij}(x, t) = \delta\lambda(x)\nabla \cdot \eta(x)\,\delta_{ij} + \delta\mu(t)[\partial_i\eta_j(x) + \partial_j\eta_i(x)] \tag{2.4}$$

The solution to (2.3) can be expressed in terms of the Green's tensor $G^0_{ij}(x)$ by

$$\eta_j(x) = \eta^0_j(x) + \sum_{l,m}\int d^3x'\, G^0_{jl}(x - x')\, V_{lm}(x')\, \eta_m(x') \tag{2.5}$$

where

$$G^0_{ij}(x - x') = \frac{1}{4\pi\rho^0\omega^2}\left\{\frac{\omega^2}{v_s^2}\delta_{ij}\frac{e^{i\omega R/v_c}}{R} - \partial_i\partial'_j\left[\frac{e^{i\omega R/v_c}}{R} - \frac{e^{i\omega R/v_s}}{R}\right]\right\} \tag{2.6}$$

$R \equiv |x - x'|$ is the distance from the inhomogeneity at x' to the point of observation x, $v_s = (\mu_0/\rho_0)^{1/2}$ is the speed of a shear (transverse) wave, and $v_c = [(\lambda_0 + 2\mu_0)/\rho_0]^{1/2}$ is the speed of a compression (longitudinal) wave. In (2.5) $\eta^0_j(x)$ is the solution to the homogeneous equation (2.1). In principle (2.5) is the complete solution to the scattering problem, since we know the analytic form of the Green's tensor and (2.5) can be iterated from a given homogeneity. In the present case we are more interested in obtaining an analytic expression for the index of refraction, albeit approximate, than in determining the properties of (2.5).

To facilitate the discussion let us introduce the Fourier transform of a function $F(x)$ by

$$\hat{F}(k) = \int d^3x\, e^{-ik\cdot x}F(x)$$

so that (2.5) can be rewritten

$$\hat{\eta}_j(k) = \hat{\eta}_j^0(k) = \sum_{l,m} \hat{G}_{jl}^0(k)[V_{lm}\hat{\eta}_m](k) \tag{2.7}$$

where if $\hat{e}_j$ is a unit vector along the j direction,

$$\hat{\eta}_j(k) = \hat{e}_j \tag{2.8a}$$

$$\hat{\eta}_j^0(k) = \hat{e}_j \delta(k - k_0) \tag{2.8b}$$

$$\hat{G}_{jl}^0(k) = \frac{1}{4\pi\rho^0\omega^2}\left\{\frac{\omega^2}{v_s^2}\frac{\delta_{jl}}{k^2 - \omega^2/v_s^2} + \hat{k}_j\hat{k}_l\left[\frac{1}{k^2 - \omega^2/v_c^2} - \frac{1}{k^2 - \omega^2/v_s^2}\right]k^2\right\} \tag{2.8c}$$

and

$$[V_{lm}\hat{\eta}_m](k) \equiv \int d^3x'\, e^{-ik\cdot x'} V_{lm}(x')\,\eta_m(x') \tag{2.8d}$$

If we multiply (2.7) on the left by a unit vector $\hat{a}_j$ and sum over j, and if $\hat{a}_j = \hat{k}_j$, then from Gubernatis *et al.*[4] we have

$$\sum_{l,m} \hat{k}_l \int e^{-ik\cdot x'} V_{lm}(x')\,\eta_m(x')\,d^3x' = 4\pi\rho^0 v_c^2 A(k)\,\rho \tag{2.9}$$

where $A(k)$ is the longitudinal, forward scattering amplitude for a single scatterer and ρ is the density of scatterers. Substituting the quantities (2.8) and (2.9) into (2.7) and simplifying yields

$$k^2 = \frac{\omega^2}{v_c^2} + \rho A(k) \tag{2.10}$$

Similarly if the initial wave is transversely polarized then $\sum_i \hat{a}_j\hat{k}_j = 0$ and[4]

$$\sum_{l,m} \hat{k}_l \int e^{-ik\cdot x'} V_{lm}(x')\,\eta_m(x')\,d^3x' = 4\pi\rho^0 v_s^2 B(k)\,\rho \tag{2.11}$$

where $B(k)$ is the transverse, forward scattering amplitude so that again by substitution into (2.7) we obtain

$$k^2 = \frac{\omega^2}{v_s^2} + \rho B(k) \tag{2.12}$$

Equations (2.10) and (2.12) can be summarized in the single equation

$$k^2 = k_0^2 + 4\pi\rho f(k) \tag{2.13}$$

where k_0 is the incident wave number of the acoustic wave, ω/v_c for a longitudinally polarized wave and ω/v_s for a transversely polarized wave. The function $f(k)$ is the associated forward scattering amplitude for a single scatterer in these two cases and is complex. For weak scattering we neglect the interference among scatterers and expand (2.13) to obtain

$$k = k_0 + 2\pi\rho Ref(k_0)/k_0 + 2\pi i\rho Imf(k_0)/k_0 \tag{2.14}$$

Using the optical theorem we relate the total cross section of the scatterer to the imaginary part of the scattering amplitude, i.e., $\sigma_t = 4\pi Imf(k_0)/k_0$. Thus the attenuation coefficient $\alpha(\omega) = 2Imk$, with $\bar{\sigma} \equiv \sigma_t/4\pi$ is

$$\alpha(\omega) = \rho\bar{\sigma} \tag{2.15}$$

We have thus shown directly from the integral equation (2.1) that if one neglects the correlations among the individual scatterings, that a transversely or longitudinally polarized ultrasonic wave propagating through an inhomogeneous elastic solid has a decay rate given by the product of the total cross section of an individual scatter and the density of scatterers. This result agrees with that of Foldy and Lax for scalar waves.

3. FRACTAL DENSITY OF SCATTERERS

To determine the total frequency dependence of the attenuation coefficient in (2.15) we utilize two facts: (1) the scattering cross section $\bar{\sigma}$ has the frequency dependence ω^s, where $s = 0$, 2, 4 from (1.1)–(1.4), and (2) the experimental observation that a high-frequency wave interacts with more scatterer than does a low-frequency wave. Therefore the density of scatterers experienced by the incident wave is frequency dependent, i.e., $\rho = \rho(\omega)$. A wave frequency ω_1, will be scattered by N_1 $(=\rho_1 V)$ defects (V is the volume of the sample and ρ_1, the density probed by the frequency ω_1) and a wave of frequency ω_2 will be scattered from N_2 defects. Thus if $\omega_2 < \omega_1$ then $\rho_2 < \rho_1$. For real constants a and b we write

$$\rho(\omega_1) = b\rho(\omega_2) = b\rho(\omega_1/a) \qquad \text{with} \quad a, b > 1 \tag{3.1}$$

which has the scaling solution

$$\rho(\omega) = \text{const}\ \omega^{\ln b/\ln a} \tag{3.2}$$

The exponent $\ln b/\ln a$ is reminiscent of a fractal dimension.[9]

Fractals are geometric objects having structure on an infinite number of scales. As an example consider a three-dimensional distribution of mass

points such that the quantity of mass $M(R)$ contained in a sphere of radius R increases with distance as

$$M(R) \propto R^F, \qquad F \leqslant 3 \tag{3.3}$$

The case $F = 3$ is the familiar situation for a uniform distribution of mass points. However, a self-similar distribution of mass points is described by a value of F less than 3. In the usual case, when the mass is within a sphere of radius R and we have no information on scales below R, then the mass is assumed to have a uniform distribution. If we now examine the sphere on a finer scale, $R' = R/a$, say, we discover that what we had considered to be a single sphere to actually consist of b smaller sphere each of radius R/a. If this process of increasing the resolution is continued *ad infinitum* we arrive at the expression (3.3) for the mass distribution with $F = \ln b/\ln a$. The quantity F is called the fractal dimension.[9]

The attenuation coefficient (2.15) can now be written

$$\alpha(\omega) = \rho(\omega)\,\bar{\sigma}(\omega) = B\omega^{\mu} \tag{3.4}$$

where B and μ are constants and μ is given by

$$\mu = s + \ln b/\ln a \tag{3.5}$$

If the material consists of grains such that $\lambda \gg D$ then the density of scatterers probed by the acoustic wave is unchanged as the frequency is increased, provided that we remain in the Rayleigh scattering domain. In this case $b = 1$ so that $\mu = s = 4$. In practice the Rayleigh domain sets in at about $\lambda \approx 10D$.[10] If we are in the scattering domain $\lambda \leqslant D$ then $s = 2$ in (3.5) and experimentally[2] $\ln b/\ln a \leqslant 2$, i.e., $b \leqslant a^2$. Thus the density of scatterers increases no more rapidly than the square of the linear scale (a^2) rather than as its cube as it would in the usual situation. This result implies that the density of scatterers is a fractal in the stochastic scattering domain. Note also that the surface of a grain can be quite rough. This roughness consists of many scales and may in part be responsible for the fractal behavior observed in the phenomenological expression (3.4).

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